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INVESTIGATION OF STABILITY OF CONSTANT LAGRANGE SOLUTIONS OF THE PLANE UNRESTRICTED THREE-BODY PROBLEM

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The problem of stability of triangular constant Lagrange solutions of the plane unrestricted three-body problem is considered in nonlinear formulation. Regions of stability for the majority of initial conditions and for formal stability are derived in the parameter plane of the problem. It is shown that when resonance relations are satisfied, Liapunov instability occurs or (in one of the resonance case) stability of finite order exists.

1. Let us consider the plane motion of three points A_1, A_2, A_3 of mass m_1, m_2, m_3, respectively, mutually attracted in conformity with Newton's law. We introduce in the plane of motion the inertial barocentric system of coordinates OXY and the Jacobi system of coordinates q_1, q_2, q_3, q_4 in which $q_1 = A_1A_2$ and q_2 and q_3 are projections of vector $\mathbf{A_0A_3}$ (A_0 is the center of mass of system A_1 and A_2) on directions parallel and perpendicular to vector $\mathbf{A_1A_2}$, and q_4 is the angle between vector $\mathbf{A_1A_2}$ and the OX-axis. Configuration of the system is completely defined by coordinates q_1, q_2, q_3. The system of differential equations defining the variation of these coordinates can be written in the canonical form [1]

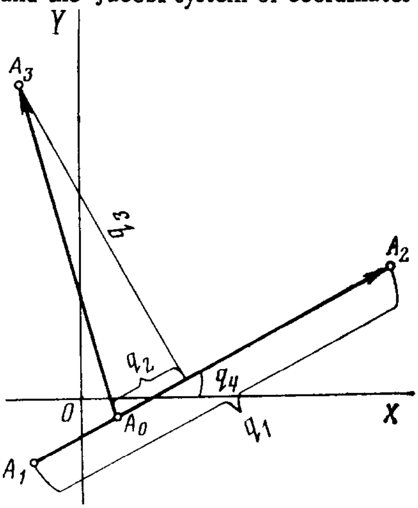

Fig. 1

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (1.1)$$

$(i = 1, 2, 3)$

$$H = \frac{m_1 + m_2}{2m_1m_2}\left[p_1^2 + \frac{1}{q_1^2}(\Gamma - q_2p_3 + q_3p_2)^2\right] +$$

$$\frac{m_1 + m_2 + m_3}{2m_3(m_1 + m_2)}(p_2^2 + p_3^2) - f\left(\frac{m_2m_3}{r_{23}} + \frac{m_1m_3}{r_{13}} + \frac{m_1m_2}{r_{12}}\right)$$

$$r_{23}^2 = \left(-\frac{m_1}{m_1 + m_2}q_1 + q_2\right)^2 + q_3^2,$$

$$r_{13}^2 = \left(\frac{m_2}{m_1 + m_2}q_1 + q_2\right)^2 + q_3^2, \quad r_{12} = q_1$$

where Γ is the area integral and f the gravitational constant.

The equations of motion in the three-body problem admit particular solutions called constant Lagrange solutions in which the three moving points constitute an invariable equilateral triangle with sides of length a [1, 2]. The triangle rotates at constant angular velocity ω about the system center of mass. For the Lagrange solution we then have $\Gamma \neq 0$ and

$$a = \frac{1}{f}\Gamma^2 \frac{M_1}{M_2^2}, \quad \omega = f^2 \frac{1}{\Gamma^3}\frac{M_2^3}{M_1}$$

$$M_1 = m_1 + m_2 + m_3, \quad M_2 = m_1m_2 + m_2m_3 + m_3m_1$$

We carry out in system (1.1) the canonical substitution of variables

$$q_i = ax_i = \frac{1}{f}\Gamma^2 \frac{M_1}{M_2^2}x_i, \quad p_i = f\frac{M_1M_2}{\Gamma}y_i \quad (i = 1, 2, 3)$$

of valency $\Gamma M_1^2 / M_2$ and introduce the new independent variable ν defined by formula

$$\frac{d\nu}{dt} = \omega = f^2 \frac{1}{\Gamma^3}\frac{M_2^3}{M_1}$$

i.e. in the constant Lagrange solution $\nu = q_4$. In new variables the equations of motion are of the form

$$\frac{dx_i}{d\nu} = \frac{\partial H^*}{\partial y_i}, \quad \frac{dy_i}{d\nu} = -\frac{\partial H^*}{\partial x_i} \quad (i = 1, 2, 3) \tag{1.2}$$

$$H^* = \frac{(m_1+m_2)M_1}{2m_1m_2}\left[y_1^2 + \frac{1}{x_1^2}\left(\frac{M_2}{M_1^2} - x_2y_3 + x_3y_2\right)^2\right] +$$

$$\frac{M_1^2}{2m_3(m_1+m_2)}(y_2^2 + y_3^2) - \frac{1}{M_1^2}\left(\frac{m_2m_3}{r_{23}} + \frac{m_1m_3}{r_{13}} + \frac{m_1m_2}{r_{12}}\right)$$

$$r_{23}^2 = \left(\frac{-m_1}{m_1+m_2}x_1 + x_2\right)^2 + x_3^2,$$

$$r_{13}^2 = \left(\frac{m_2}{m_1+m_2}x_1 + x_2\right)^2 + x_3^2, \quad r_{12} = x_1$$

The Lagrange solutions correspond to equilibrium positions of system (1.2)

$$x_1^\circ = 1, \quad x_2^\circ = \frac{m_1 - m_2}{2(m_1+m_2)}, \quad x_3^\circ = \pm\frac{\sqrt{3}}{2} \tag{1.3}$$

$$y_1^\circ = 0, \quad y_2^\circ = \mp\frac{m_3(m_1+m_2)}{M_1^2}\frac{\sqrt{3}}{2}, \quad y_3^\circ = \frac{m_3(m_1-m_2)}{M_1^2}$$

All subsequent analysis relates to $x_3^\circ = +\sqrt{3}/2$, since it is also valid for $x_3^\circ = -\sqrt{3}/2$.

We introduce the notation

$$\alpha = \frac{m_3}{m_1+m_2+m_3}, \quad \beta = \frac{m_1 - m_2}{m_1+m_2},$$

$$\gamma = \frac{m_1m_2}{m_1^2 + m_1m_2 + m_2^2} = \frac{1-\beta^2}{3+\beta^2}$$

and carry out in system (1.2) the canonical substitution of variables

$$x_1 = X_1, \qquad y_1 = Y_1(\beta^2+3)(1-\alpha)$$

$$x_2 = \beta X_2 - \sqrt{3}X_3, \quad y_2 = (\beta Y_2 - \sqrt{3}Y_3)(1-\alpha)$$

$$x_3 = \sqrt{3}X_2 + \beta X_3, \quad y_3 = (\sqrt{3}Y_2 + \beta Y_3)(1-\alpha)$$

of valency $(\beta^2 + 3)(1 - \alpha)$. In the new variables the equations of motion are of the form

$$\frac{dX_i}{d\nu} = \frac{\partial K}{\partial Y_i}, \quad \frac{\partial Y_i}{\partial \nu} = -\frac{\partial K}{\partial X_i} \quad (i = 1, 2, 3) \tag{1.4}$$

$$K = \frac{2}{\gamma}\left[Y_1^2 + \frac{1}{X_1^2}\left(\frac{\gamma}{4} + \frac{\alpha}{4} - X_2Y_3 + X_3Y_2\right)^2\right] + \tag{1.5}$$

$$\frac{1}{2\alpha}(Y_2^2 + Y_3^2) - \frac{(1-\alpha)\gamma}{4X_1} - \frac{\alpha}{2(\beta^2+3)}\left(\frac{1-\beta}{r_1} + \frac{1+\beta}{r_2}\right)$$

$$r_{1,2}^2 = \frac{(1\pm\beta)^2}{4}X_1^2 - (\beta\pm 1)X_1(\beta X_2 - \sqrt{3}X_3) +$$

$$(\beta^2 + 3)(X_2^2 + X_3^2)$$

and the Lagrange solution (1.3) is of the form

$$X_1^\circ = 1, \quad X_2^\circ = 1/2, \quad X_3^\circ = 0$$
$$Y_1^\circ = 0, \quad Y_2^\circ = 0, \quad Y_3^\circ = \alpha / 2 \tag{1.6}$$

As in [1, 2], we assume the constant Lagrange solution to be stable, if in the perturbed motion the triangle formed by points A_1, A_2, A_3, always deviates as little as desired from the initial equilateral form. In such formulation the problem of the Lagrange solution stability is equivalent to that of stability of the equilibrium position (1.6) of system (1.4) with respect to perturbations of coordinates and momenta.

Routh and Joukowskii [3] obtained the necessary conditions of the Lagrange solution stability in the case of an arbitrary power law of attraction. It was shown in [4] that these necessary conditions are not sufficient; instability was proved in the case of certain values of parameters.

The contemporary development of the theory of stability of Hamiltonian systems enables us to analyze more fully the stability of the Lagrange solutions of the classic plane circular unrestricted three-body problem. And this is the aim of the present work.

2. We determine the perturbed motion in the neighborhood of the equilibrium position (1.6) using the coordinates Q_i and momenta P_i defined by formulas

$$X_i = X_i^\circ + Q_i, \quad Y_i = Y_i^\circ + P_i \quad (i = 1, 2, 3) \tag{2.1}$$

We represent the Hamiltonian (1.5) in the form of series

$$K = K_2 + \ldots + K_n + \ldots \tag{2.2}$$

where K_n is a homogeneous polynomial of power n in Q_i and P_i $(i = 1, 2, 3)$. The terms of series (2.2) which are subsequently required are of the form

$$K_2 = \frac{2}{\gamma}P_1^2 + \frac{1}{2\alpha}P_2^2 + \left(\frac{1}{2\alpha} + \frac{1}{2\gamma}\right)P_3^2 + Q_3P_2 +$$
$$\left(\frac{\alpha}{\gamma} - 1\right)Q_2P_3 + Q_1P_3 + \left(\frac{\gamma}{8} + \frac{9}{32}\alpha\gamma\right)Q_1^2 +$$
$$\left(\alpha - \frac{9}{8}\alpha\gamma\right)Q_1Q_2 - \frac{3\sqrt{3}}{8}\alpha\beta\gamma Q_1Q_3 +$$
$$\left(\frac{\alpha^2}{2\gamma} - \alpha + \frac{9}{8}\alpha\gamma\right)Q_2^2 + \frac{3\sqrt{3}}{4}\alpha\beta\gamma Q_2Q_3 + \left(\frac{\alpha}{2} - \frac{9}{8}\alpha\gamma\right)Q_3^2$$

$$K_3 = -\frac{1}{\gamma}Q_1P_3^2 - 2Q_1Q_3P_2 - 2\left(\frac{\alpha}{\gamma} - 1\right)Q_1Q_2P_3 - \frac{3}{2}Q_1^2P_3 -$$

$$\frac{2}{\gamma}Q_3P_2P_3-\frac{2\alpha}{\gamma}Q_2Q_3P_2+\frac{2}{\gamma}Q_2P_3^2+\frac{2\alpha}{\gamma}Q_2^2P_3-$$
$$\left[\frac{(\alpha+1)\gamma}{4}+\frac{7}{128}\alpha\gamma(\beta^2+1)\right]Q_1^3+\left[\frac{\alpha\gamma}{64}(21\beta^2+9)-\right.$$
$$\left.\frac{3}{2}\alpha\right]Q_1^2Q_2-\left[\frac{\alpha^2}{\gamma}+\frac{3}{32}\alpha\gamma(7\beta^2-33)\right]Q_1Q_2^2-$$
$$\frac{9\sqrt{3}}{32}\alpha\beta\gamma Q_1^2Q_3+\frac{27\sqrt{3}}{8}\alpha\beta\gamma Q_1Q_2Q_3+\frac{3}{32}\alpha\gamma(11\beta^2-21)\times$$
$$Q_1Q_3^2-\left[\frac{7}{16}\alpha(\beta^2+3)+\frac{45}{8}\alpha\gamma-\frac{15}{4}\alpha\right]Q_2^3-$$
$$\frac{45\sqrt{3}}{8}\alpha\beta\gamma Q_2^2Q_3-\left[\frac{3}{4}\alpha(\beta^2+3)-\frac{45}{16}\alpha\gamma(3-\beta^2)\right]Q_2Q_3^2+$$
$$\frac{15\sqrt{3}}{8}\alpha\beta\gamma Q_3^3$$

$$K_4=\frac{2}{\gamma}Q_3^2P_2^2+\frac{2}{\gamma}Q_2^2P_3^2+\frac{3}{2\gamma}Q_1^2P_3^2+3Q_1^2Q_3P_2+$$
$$3\left(\frac{\alpha}{\gamma}-1\right)Q_1^2Q_2P_3+2Q_1^3P_3-\frac{4}{\gamma}Q_2Q_3P_2P_3+$$
$$\frac{4}{\gamma}Q_1Q_3P_2P_3+\frac{4\alpha}{\gamma}Q_1Q_2Q_3P_2-\frac{4}{\gamma}Q_1Q_2P_3^2-\frac{4\alpha}{\gamma}Q_1Q_2^2P_3+$$
$$\left[\frac{3}{8}\gamma+\frac{\alpha\gamma}{4}+\frac{37}{2048}\alpha\gamma(3\beta^2+1)\right]Q_1^4+\left[2\alpha+\frac{\alpha\gamma}{256}\times\right.$$
$$\left.(\beta^2+75)\right]Q_1^3Q_2+\frac{\sqrt{3}}{256}\alpha\beta\gamma(25\beta^2+99)Q_1^3Q_3+$$
$$\left[\frac{3\alpha^2}{2\gamma}-\frac{\alpha\gamma}{256}(339\beta^2+369)\right]Q_1^2Q_2^2-\frac{3\sqrt{3}}{128}\alpha\beta\gamma(25\beta^2+3)\times$$
$$Q_1^2Q_2Q_3+\frac{\alpha\gamma}{256}(327\beta^2+333)Q_1^2Q_3^2+\frac{45}{64}\alpha\gamma(5\beta^2-9)\times$$
$$Q_1Q_2^3+\frac{15\sqrt{3}}{64}\alpha\beta\gamma(5\beta^2-57)Q_1Q_2^2Q_3-\frac{45}{64}\alpha\gamma(19\beta^2-15)\times$$
$$Q_1Q_2Q_3^2+\frac{\alpha}{128}(337\beta^2+2520\gamma-849)Q_2^4-$$
$$\frac{15\sqrt{3}}{64}\alpha\beta\gamma(3\beta^2-15)Q_1Q_3^3-\frac{5\sqrt{3}}{32}\alpha\beta\gamma(5\beta^2-153)Q_2^3Q_3+$$
$$\frac{3\alpha}{64}[64(\beta^2+3)+105\gamma(5\beta^2-9)]Q_2^2Q_3^2+\frac{15\sqrt{3}}{32}\alpha\beta\gamma\times$$
$$(3\beta^2-47)Q_2Q_3^3-\left[\frac{3}{8}\alpha(\beta^2+3)+\frac{45}{128}\alpha\gamma(13\beta^2-17)\right]Q_3^4$$

The determining equation is of the form

$$(\sigma^2+1)(\sigma^4+\sigma^2+k)=0 \qquad (2.3)$$
$$k=\frac{27}{4}\left[\alpha(1-\alpha)+(1-\alpha)^2\frac{1-\beta^2}{4}\right]=\frac{27}{4}\frac{m_1m_2+m_2m_3+m_3m_1}{(m_1+m_2+m_3)^2}$$

and the necessary condition of stability is

$$k<1/4 \qquad (2.4)$$

The following interpretation of condition (2.5) conforms to [5] (Fig. 2). In the plane of the equilateral triangle A_1, A_2, A_3 to each set of three m_1, m_2, m_3 corresponds a single point that denotes the system center of mass which lies on the circumference of radius $\rho(k)$, whose center lies at the triangle geometric center, and

$$\rho(k) = \rho_0 \sqrt{1-4k/9} \tag{2.5}$$

where ρ_0 is the radius of the circumscribed circle. The position of the centeı of mass in one of the three regions outside the circle of radius

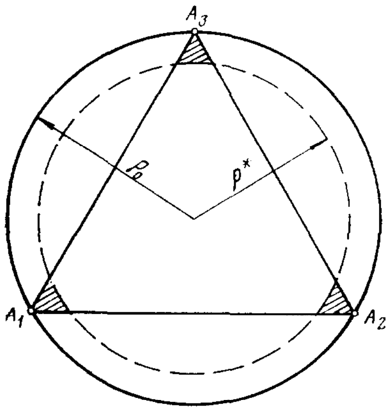

Fig. 2

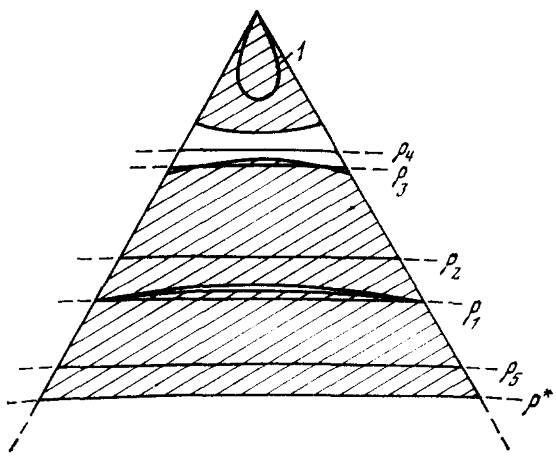

Fig. 3

ρ^*, where

$$\rho^* = \rho(1/4) = \rho_0 \sqrt{8/9}$$

corresponds in Fig. 2 to condition (2.4).

Parameter α is proportional to the distance from the center of mass to the straight line A_1A_2, and $\alpha = 1$ corresponds to the position of the center of mass at point A_3 (i.e. $m_1 = m_2 = 0$).

The canonical linear normalizing transform

$$(Q_1, Q_2, Q_3, P_1, P_2, P_3) = (Q_1^*, Q_2^*, Q_3^*, P_1^*, P_2^*, P_3^*) N$$

is obtained with the use of the algorithm set forth in [6]. For the rows N_i of the simplistic Matrix N we obtain the expressions

$$N_1 = \frac{c_1}{\lambda_1}(2, 1, 0, 0, 0, -\alpha), \quad N_4 = c_1\left(0, 0, 0, \frac{\gamma}{2}, \alpha, 0\right)$$

$$N_j = \frac{c_j}{\lambda_j}\left(\frac{\varkappa_j}{\gamma}, -\frac{\varkappa_j}{2\alpha}, -\frac{3\sqrt{3}}{2}\beta\frac{(\alpha+\gamma)^2}{\alpha\gamma}, 0, \frac{3\sqrt{3}}{2} \times \right.$$

$$\left.\beta\frac{(\alpha+\gamma)^2}{\gamma}, \frac{\varkappa_j}{2} - 9\alpha - 9\gamma\right)$$

$$N_{3+j} = c_j\left(0, 0, -\frac{4}{\alpha} - \frac{4}{\gamma}, \frac{\varkappa_j}{4}, 4 + \frac{4\alpha}{\gamma} - \frac{\varkappa_j}{2}, \right.$$

$$\left. -\frac{3\sqrt{3}}{2}\beta(\alpha+\gamma)\right)$$

$$c_j^2 = \frac{\lambda_j}{2}\frac{\alpha\gamma}{(\alpha+\gamma)(\lambda_j^2 - {}^1/_2)\varkappa_j}, \varkappa_j = 9\alpha + 9\gamma + 4\lambda_j^2 \quad (j = 2, 3)$$

$$c_1^2 = \frac{1}{\alpha+\gamma}, \quad \lambda_1 = 1, \quad \lambda_{2,3} = \pm\sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - k}}$$

The Hamiltonian (2.2) in terms of variables Q_i^*, P_i^* $(i = 1, 2, 3)$ is of the form

$$K^* = \frac{\lambda_1}{2}(Q_1^{*2} + P_1^{*2}) + \frac{\lambda_2}{2}(Q_2^{*2} + P_2^{*2}) + \frac{\lambda_3}{2}(Q_3^{*2} + P_3^{*2}) + \ldots$$
$$+ K_n^* + \ldots$$

We pass to "polar" coordinates using formulas

$$Q_i^* = \sqrt{2r_i}\sin\varphi_i, \quad P_i^* = \sqrt{2r_i}\cos\varphi_i \quad (i = 1, 2, 3)$$

and obtain

$$K^* = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 + \ldots + K_n^* + \ldots \tag{2.6}$$

where K_n^* is a function homogeneous with respect to r_i $(i = 1, 2, 3)$ of order $n/2$.

3. For the investigation of stability we carry out further (nonlinear) normalization of the system with Hamiltonian (2.6), essentially following the investigation scheme presented in [6].

The process of nonlinear normalization depends to a considerable extent on the presence in the system of resonances, i.e. of relations of the form

$$n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 = 0 \tag{3.1}$$

where n_1, n_2, n_3 are integers. The quantity $|n_1| + |n_2| + |n_3|$ is called the order of a resonance. When resonances up to fourth order are absent, the Hamiltonian (2.6) can be brought to the form

$$K^* = K_2^* + K_4^* + \ldots \tag{3.2}$$
$$K_2^* = \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3$$
$$K_4^* = c_{200}R_1^2 + c_{110}R_1R_2 + c_{101}R_1R_3 + c_{020}R_2^2 + c_{011}R_2R_3 + c_{002}R_3^2$$

by the canonical nonlinear transformation $\varphi_i, r_i \rightarrow \Phi_i, R_i$ $(i = 1, 2, 3)$, with the coefficients of form K_4^* dependent on the problem parameters k and α but independent of the method of reducing the Hamiltonian to the normal form (3.2).

In the linear stability region of this problem the following resonance relations of up to fourth order are possible:

$$\begin{aligned}
&\lambda_1 + 2\lambda_3 = 0, \quad k = k_1 = 0.1875 \\
&\lambda_2 + 2\lambda_3 = 0, \quad k = k_2 = 0.16 \\
&\lambda_1 + 3\lambda_3 = 0, \quad k = k_3 = 8/81 = 0.09876\ldots \\
&\lambda_2 + 3\lambda_3 = 0, \quad k = k_4 = 0.09 \\
&\lambda_1 - 2\lambda_2 - \lambda_3 = 0, \quad k = k_5 = 0.2304
\end{aligned} \tag{3.3}$$

The curves along which the resonance relations (3.3) are satisfied are shown in

Fig. 3 for one of the three symmetric regions appearing in Fig. 2. Since frequencies λ_i $(i = 1, 2, 3)$ depend only on parameter k, these curves are arcs of circles of radii $\rho_j = \rho(k_j)$ $(j = 1, \ldots, 5)$ calculated by formula (2.5).

The Arnol'd — Moser theorem [7] holds for the canonical system with Hamiltonian (3.2); if at least one of the determinants

$$D_3 = \det \left\| \frac{\partial^2 K_4^*}{\partial R_i \partial R_j} \right\|, \quad D_4 = \det \left\| \begin{matrix} \dfrac{\partial^2 K_4^*}{\partial R_i \partial R_j} & \lambda_i \\ \lambda_j & 0 \end{matrix} \right\| \tag{3.4}$$

is nonzero, the equilibrium position is stable for the majority of initial conditions.

Coefficients of the normal form (3.2) and determinants (3.4) were computed outside of resonance curves (3.3) on a computer using the algorithms presented in the paper of A. P. Markeev and A. G. Sokol'skii under the title: Certain computing algorithms of Hamiltonian systems normalization (preprint of the Inst. of Problems of Mechanics, No. 31, 1976). The closed curve *1* in Fig. 3 represents points at which $D_4 = 0$; inside that curve $D_4 < 0$ and outside it $D_4 > 0$. As shown by the computations, $D_3 \neq 0$ at all points of that curve, hence for all $k < 0.25$, except for $k = k_i$ $(i = 1, \ldots, 5)$, we have stability for the majority of initial conditions.

4. One more type of stability, viz. formal stability [8], which means that instability cannot be detected by taking into account any finite number of terms of expansion of the Hamiltonian is considered here.

For a system with Hamiltonian (3.2) formal stability occurs when the system [6]

$$K_2^* = 0, \qquad K_4^* = 0$$

has no nontrivial solutions in the region $R_i \geqslant 0$ $(i = 1, 2, 3)$. Since $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, hence the form

$$K_4^* \left(R_1, R_2, -\frac{\lambda_1}{\lambda_3} R_1 - \frac{\lambda_2}{\lambda_3} R_2\right) = AR_1^2 + BR_1R_2 + CR_2^2 \tag{4.1}$$

$$A = c_{200} - \frac{\lambda_1}{\lambda_3} c_{101} + \frac{\lambda_1^2}{\lambda_3^2} c_{002}, \quad C = c_{020} - \frac{\lambda_2}{\lambda_3} c_{011} + \frac{\lambda_2^2}{\lambda_3^2} c_{002}$$

$$B = c_{110} - \frac{\lambda_2}{\lambda_3} c_{101} - \frac{\lambda_1}{\lambda_3} c_{001} + 2 \frac{\lambda_1 \lambda_2}{\lambda^2_3} c_{002}$$

of fixed sign in region $R_1 \geqslant 0$, $R_2 \geqslant 0$. This is possible if $B^2 - 4AC < 0$ or A, B, and C are of the same sign. It can be ascertained that

$$B^2 - 4AC = \lambda_3^{-2} D_4$$

hence inside curve *1* (Fig. 3) we have formal stability, while for form (4.1) to be of fixed sign outside it, it is necessary that A, B, C are of the same sign. The results of numerical analysis appear in Fig. 3. Formal stability exists in the shaded regions, except on resonance curves.

5. Let us investigate stability of equilibrium position (1.6) of system (1.4) in the resonance cases (3.3).

We begin by considering the first four cases of (3.3) for which in formulas (3.1) $n_i \geqslant 0$ $(i = 1, 2, 3)$. If $n_1 + n_2 + n_3 = 3$, the normal form of Hamiltonian

(2.6) is

$$K^* = K_2^* + a_{n_1n_2n_3}\sin(n_1\Phi_1 + n_2\Phi_2 + n_3\Phi_3) R_1^{n_1/2} R_2^{n_2/2} R_3^{n_3/2} + \ldots \quad (5.1)$$

where K_2^* is of the form (3.2). If $a_{n_1n_2n_3}^* \neq 0$, the equilibrium position is Liapunov unstable.

Computations have shown that $a_{102} \neq 0$ when $k = k_1$ and $a_{012} \neq 0$ when $k = k_2$ for all α from the region of linear stability, hence when $k = k_1$ and $k = k_2$ the equilibrium position (1.6) is Liapunov unstable.

When relations (3.1) for $n_1 + n_2 + n_3 = 4$ are satisfied, the normal form of Hamiltonian (2.6) is

$$K^* = K_2^* + W(R_1, R_2, R_3) + a_{n_1n_2n_3}\sin(n_1\Phi_1 + n_2\Phi_2 + n_3\Phi_3) R_1^{n_1/2} R_2^{n_2/2} R_3^{n_3/2} + \ldots$$

where K_2^* is of the form (3.2) and $W(R_1, R_2, R_3)$ is the quadratic form of R_1, R_2, R_3. If

$$|a_{n_1n_2n_3}| (n_1^{n_1} n_2^{n_2} n_3^{n_3})^{1/2} > |W(n_1, n_2, n_3)| \quad (5.2)$$

the equilibrium position is Liapunov unstable, while when the inequality is of the opposite sign, we have stability of the "truncated" system.

Numerical analysis has shown that for resonances determined by the third and fourth system of equalities (3.3), the inequality (5.2) is valid for all α from the linear stability region, hence when $k = k_3$ and $k = k_4$ the equilibrium position (1.6) is unstable.

The last of resonances (3.3) differs from those already considered by that in its case a change of sign takes place among some of the numbers n_1, n_2, n_3. Then the system with the Hamiltonian

$$K^* = K_2^* + W(R_1, R_2, R_3) + a_{121}\sin(\Phi_1 - 2\Phi_2 - \Phi_3) \times R_1^{1/2} R_2 R_3^{1/2}$$

reduced to second order terms with respect to R_i $(i = 1, 2, 3)$ has the fixed sign integral

$$L = 3R_1 + R_2 + R_3$$

However, the effect of taking into account terms of higher order is unknown, since when $k = k_5$ we have $\lambda_1 = 1$, $\lambda_2 = 0.8$, $\lambda_3 = -0.6$, and the system admits besides the indicated resonance, other resonance relations, for instance two resonances of fifth order

$$\lambda_1 + \lambda_2 + 3\lambda_3 = 0, \quad 2\lambda_1 - \lambda_2 + 2\lambda_3 = 0$$

Hence, when $k = k_5$ it is possible to assert the stability of the equilibrium position (1.6) only when resonances of order not higher than the fourth with respect to coordinates and momenta are taken into account.

The above analysis makes it possible to formulate the basic conclusion as follows.

In the stability region $k < 0.25$ in the first approximation of constant Lagrange solutions of the plane unrestricted three-body problem there are four curves that correspond to resonance values of parameter k, along which Liapunov instability occurs. Along the fifth resonance curve stability of the finite order is present.

Outside of the five resonance curves in the stability region in the first approximation we have stability for the majority of initial conditions.

Regions in which the constant Lagrange solutions are formally stable have been determined in the first approximation stability region. Outside of these regions (obviously, inside the first approximation stability region) the presence of formal stability of Lagrange solutions can also be confirmed (possibly excluding a finite number of curves and points in the parameter plane) by analyzing stability, taking into account forms K_5 and K_6 in expansion (2.2) (see [6]). However such investigation which takes into account therms of the fifth, sixth, and possibly higher order may be omitted, since the described above investigation provides a fairly comprehensive representation of formal stability of Lagrange solutions.

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